MISSISSIPPI
STATE GEOLOGICAL SURVEY

WILLIAM CLIFFORD MORSE, Ph.D.
Director

BULLETIN 70

RATE OF DEPLETION OF WATER-BEARING SANDS

By

FREDERIC HARTWELL KELLOGG, Ph.D.

CONTRIBUTION OF THE DEPARTMENT
OF CIVIL ENGINEERING

UNIVERSITY, MISSISSIPPI
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MISSISSIPPI GEOLOGICAL SURVEY

LETTER OF TRANSMITTAL

Office of the Mississippi Geological Survey
University, Mississippi
June 26, 1950

To His Excellency

Governor Fielding Lewis Wright, Chairman, and

Members of the Geological Commission

Gentlemen:

Herewith is Bulletin 70, Rate of depletion of water-bearing sands by Frederic Hartwell Kellogg.

Although it is an extremely technical paper, it is of the utmost value in setting aright those who compute our available ground-water supply.

Very truly yours,

William Clifford Morse,

Director and State Geologist

RATE OF DEPLETION OF WATER-BEARING SANDS

FREDERIC HARTWELL KELLOGG

At the present time, much is being heard of the serious depletion of groundwater supplies in various parts of the country. This is a particular problem in certain densely-populated sections along the east and west coasts. In the lower Mississippi Valley, and in the State of Mississippi in general, reserves of usable groundwater are still ample. In considering the potentialities for growth and industrial expansion in these regions, therefore, it is possible to avoid some of the mistakes that have led to such serious consequences in more populous regions.

The vast amounts of underground water stored in the sands of Mississippi constitute one of the most valuable of the state's natural resources. In them, nature furnishes the water purification plant. The water is not only potable and safe, but is so free of dissolved solids that, for years, mechanics and filling station operators have been using it in storage batteries with no harmful results whatsoever. The sands not only store large quantities of water, but, since they outcrop in regions of fairly high rainfall, storage can be maintained in spite of rather high withdrawals by pumping. This is in striking contrast to the conditions in the west, where low rainfall limits the recharging capacity.

These advantages are also found in such areas as Long Island and Baltimore, which are, nevertheless, faced with serious water shortage problems. To avoid or circumvent such problems, there has been evolved over the past thirty years that relatively young branch of science known as Groundwater Hydrology. The techniques and methods of analysis developed during this evolution are well known, and may be studied in the various Water Supply Papers of the U. S. Geological Survey. On the basis of studies in this line, the serious groundwater conditions now encountered in many areas have been predicted for a long time. These predictions received too little attention.

There is one major fault with plans based on present practice in Groundwater Hydrology. It lies in the fact that rate of depletion can only be estimated by actually depleting the sands through pumping. In the past five or six years, a great deal of attention has been concentrated on analytical methods of making such estimates. Admitting that analytical methods require an over-simplification of the complex conditions found in nature, they furnish a guide to judgement in interpreting statistics that would otherwise be meaningless or even misleading. Therefore, the effort expended in this direction appears certainly worthwhile.

It is frequently stated that the science of groundwater hydrology has been completely altered and reorganized during the past few years, due to research on the rates of depletion and recharge of sands. While this is undoubtedly an overstatement, it is certain that such research constitutes one of the most vital and promising fields in the study of groundwater. If, therefore, the studies made in this field to date should be based on a wrong premise, the effect on the progress of groundwater studies as a whole would indeed be serious. It is the purpose of this paper to call attention to what appears to be a fundamental fallacy in analyzing depletion and recharge rates, and to suggest an alternative method of attack such as is used under similar conditions in flow of oil through sands.

Flow of groundwater through sand may occur in either the *steady state* or the *unsteady state*. In steady state flow, all water seeping out of the sand is immediately replaced by an equal amount flowing into it. An illustration of such flow is afforded by a sand which outcrops in a river. Water pumped out of a well penetrating this sand at some distance from the river is immediately replaced by water entering the sand from the river. In this case, the river is called the *source*, while the well is known as a *sink*. In unsteady state flow, the volume of water seeping out of the sand is not equal to that flowing into it. If inflow is greater than outflow, the sand is recharged; if outflow is greater than inflow, the sand is depleted or drained.

It has long been established that steady state flow of groundwater through sand can be studied analytically by methods analogous to those used for steady state flow of heat through solids. Steady state flow of heat is that in which all heat emanating from the solid is replaced by an outside source, as in flow through the walls of an oven the inside of which is held at a constant temperature. The modern theories of rates of depletion and recharge of sands are based on the postulate that unsteady state groundwater flow is analogous to unsteady state flow of heat.[1] In unsteady state flow of heat, outflow is either greater than inflow (cooling) or less than inflow (heating). Field investigations have indicated that, at least within certain limits, this postulate is correct.[2], [3]

Assuming the validity of the basic postulate, it remains to be seen whether the premises formulated from it are likewise correct. In order to check on their validity, it is necessary to state, first, the premises on which analysis of heat flow are founded, and secondly, the analogous premises to be established for groundwater flow.

Flow of heat is governed by the *Biot-Fourier Law,* which states that the velocity of heat transmission is directly proportional to the temperature gradient (change of temperature per unit of length of the body through which heat is flowing). As an analogy, it has been known for over a hundred years that the velocity of water seeping through sand is directly proportional to the hydraulic gradient (change in hydraulic head per unit of length of the sand body). This relationship is known as *Darcy's Law.* Where temperature difference is the driving force producing heat flow, difference in head is the driving force producing groundwater flow.

The analogy between heat flow and groundwater percolation as described thus far may be expressed symbolically as follows:

Heat	Groundwater	
$V=K_h\frac{\Delta\theta}{\Delta l}$	$V=-K_w\frac{\Delta h}{\Delta l}$	(1)

where V is velocity, K_h is a constant for heat flow known as the *coefficient of heat transmission,* θ is temperature (Δ is temper-

1. Thies: "The Relation Between the Lowering of the Piezometric surface and the Rate and Duration of Discharge of a Well Using Groundwater Storage." *American Geophysical Union Transactions,* 1935, pp. 519-524.

2. Wenzel: "Methods of Determining Permeability" *U. S. Geological Survey Water Supply Paper No. 887,* Washington, 1942, pp. 87-89, 146-147.

3. Kellogg: "Investigation of Drainage Rates Affecting Stability of Earth Dams", *Trans.* A.S.C.E., *113,* 1261-1309, 1948.

ature increment or decrement), l is length (Δl is the length required for a change in temperature of $\Delta\theta$), K_w is a constant for groundwater flow known as the *permeability coefficient,* and h is hydraulic head.

Now in any dynamic problem of this nature, the Law of Conservation of Matter must apply, *i.e.,* matter (or energy) can be neither created nor destroyed. For steady state flow, since all outflow is replaced by inflow, the difference between flow out of a small element of the transmitting body (heat conductor or sand) and flow into it must be zero. Expressed analytically, the net difference between inflow and outflow for a unit volume element, or net flux for an increment of time dt is

$$\nabla v dt = o \tag{2}$$

For the unsteady state, the net flux is not equal to zero. Therefore, some other analytical expression must be equal to $\nabla v dt$ to express mathematically, the Law of Conservation of Matter. In heat flow, *specific heat,* a_h, may be considered as the quantity of heat required to raise one gram of conducting medium one degree Centigrade. Hence, if ρ is density of the medium, in grams per cubic centimeter, the quantity of heat required to raise a cubic centimeter of the medium one degree is $a_h\rho$, and the quantity required to raise this volume $\Delta\theta$ degrees is $a_h\rho\Delta\theta$. If $\Delta\theta$ be reduced to infinitesimal value, we have, then, a net flux of $a_h\rho\, d\theta$, or

$$v dt = a_h \rho d\theta \tag{3}$$

Returning to equation (1), reducing the increments $\Delta\theta$ and Δl to infinitesimal value, and substituting into equation (3),

$$K_h \nabla^2 \theta dt = a_h \rho d\theta \tag{4}$$

Equation (4) is the basic equation for the classical analyses of flow of heat in the unsteady state. While neither a_h nor ρ is truly constant, both can be assumed so with little error in analytical results.

To derive an analogous relationship for groundwater flow, we can again say that, for the steady state, the net flux is zero, or $\nabla v dt = o$. For the unsteady state, we must find another expression for net flux, to which $\nabla v dt$ can be equated. This must indicate the quantity of water that can be drained from (or forced into) a unit volume of sand due to a given change in hydraulic head

To determine this, saturated sand samples have been subjected to various differences in head at top and bottom, and the quantity of flow for each difference plotted against the corresponding head difference.[4] The result was a curve somewhat similiar to the probability curve, the equation for which is

$$q_w = \frac{A}{\sqrt{\pi}} e^{-A^2h^2} \qquad (5)$$

where A is a constant depending on the sand, and expressing the height and slope of the curve. Then the change in quantity per unit change of head is the first derivative of equation (5), or

$$\frac{dq_w}{dh} = \frac{-2A^3h}{\sqrt{\pi}} e^{-A^2h^2} \qquad (6)$$

Then the equation for unsteady state groundwater flow would be, by substituting equation (1) into the expression $\nabla V dt$,

$$K_w \nabla^2 h dt = \frac{2A^3h}{\sqrt{\pi}} e^{-A^2h^2} dh \qquad (7)$$

Equation (7) is quite different from equation (4) and indicates that there is no analogy between heat flow and groundwater flow. The right hand side of equation (4) indicates that there is a straight line relationship between quantity of heat and temperature, *i.e.*

$$dq_h = a_h \rho d\theta$$
$$qh = a_h \rho \theta + \text{constant}$$

The right hand side of equation (7) indicates the probability curve relationship of equation (5). Therefore, there cannot be even an approximate analogy between heat flow and groundwater flow unless the probability curve of equation (5) can be approximated by a straight line. Within limits, this can be done, as will be demonstrated in the following discussion.

Let us consider only drainage of a sand, since recharge is governed by principles that are clearly similar. The amount of water that a sand can hold is determined by the total volume of its pore spaces. The sizes of individual pore spaces vary considerably, according to the variety of sizes and shapes of grains.

4. Kellogg, Op. Cit., p. 1273, Fig. 6

At the top of the water, *i.e.,* where water is in contact with air, capillary forces tend to resist the flow of water out of the pores. The smaller an individual pore space, the greater is this resistance. In general, the pore spaces at least partially intercommunicate, so that the capillary force at the top of the water resists the weight of a column of water extending through the interconnected pore spaces to the point of drainage. Figure 1 shows a schematic diagram of a sand column draining vertically, in which

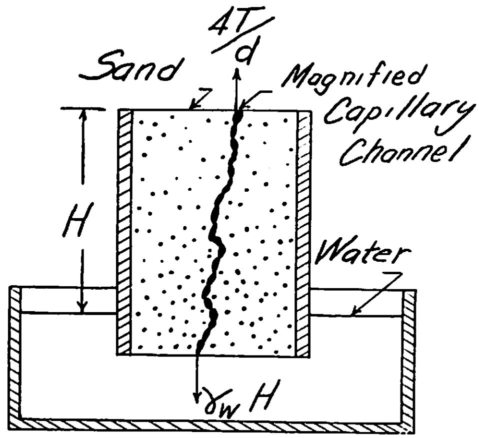

Figure 1.—Vertical Drainage of Sand Column

the unit capillary force, 4T/d (where T is surface tension, and d is diameter of pore space) just balances the unit weight of the water column, $\rho_w H$ (where ρ_w is unit weight of water and H is net height of water column). In such a condition, no drainage will occur, and the pore water is held in the sand by the capillary force. The smaller the pore diameter, d, the greater is this force opposing drainage. At the top of the sand are a virtually infinite number of pore spaces of various diameters, d. Now, if an external force $\rho_w \Delta h$ be applied at the top of the sample, as by additional air pressure, the equilibrium of the water column shown in Figure 1 is overcome, and the pore water flows downward. However, there will be other pore spaces of much smaller diameter. In these, the force 4T/d is greater than the sum of the external force $\rho_w \Delta h$ plus the weight of the water column $\rho_w H$. In these, there is no flow, but only an increase in the curvature of the meniscus at the top of the water column. A still higher external force must be applied in order to start flow in these columns. Hence, a curve plotting volume of water drained from the sand against pressure required for this drainage is a measure of the volume of pores of successively greater diameters, as measured by the successively greater external pressure required to overcome capillary resistance.

For many years, it has been customary to judge the rate of flow of groundwater through sand, as indicated by the permeability coefficient of equation (1), by an *effective size* of grain. This tactitly assumes that grain diameter is a measure of pore

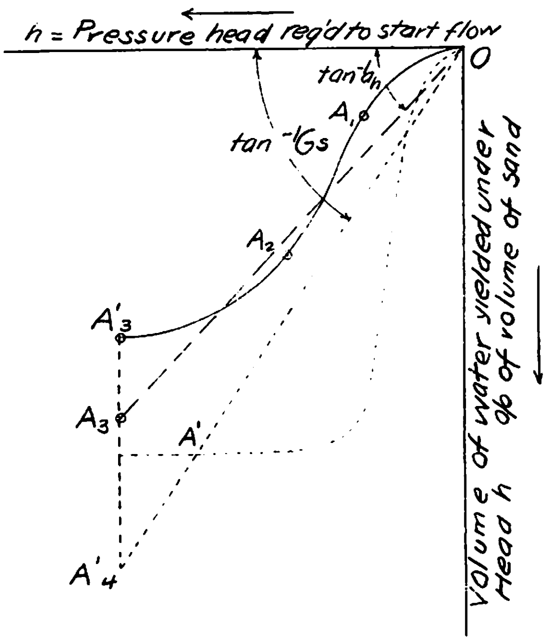

Figure 2.—Yield-Pressure Curves

diameter. Such a measure must be at least approximately correct, since the effective size method of estimating sand permeability has proved satisfactory when properly used. If then, grain size is a measure of pore diameter, and a curve showing yield plotted against pressure required to obtain this yield is a measure of pore volume, then the latter curve should resemble a mechanical analysis curve. Now, as Hazen has indicated long ago,[5] mechanical

5. Hazen; Some physical properties of sand and gravels: *Annual Report Mass. State Board of Health* 1893.

analysis curves of sands conform approximately to the probability curve. So do the yield pressure curves. Therefore, the relationship expressed by equation (7) is simply what could be expected.

To follow the similarity between yield-pressure curves and mechanical analysis curves still further, a certain section of each tends to be flat and approximate a straight line, as shown by section A_1A_2 in Figure 2. However, where the straight line continues to A_3, the curves deviate, so that grain sizes approach infinitesimal values, while external pressures required to overcome the capillary forces in the small pore spaces between these minute grains approach infinite values. This is shown at A'_3 on the Figure. If the pressure available for draining a sand (for instance, heads made available in drawing down a well by pumping) do not appreciably exceed those falling on the straight line position of the yield pressure curve, then we can approximate the probability curve by a straight line such as OA'_3 in Figure 2, having the equation

$$q_w = a_w h$$

$$dq_w = a_w dh \qquad (8)$$

Then a_w can be defined as the quantity of water that can be drained from a unit volume of sand by a unit of head difference, and designated as *coefficient of drainage*. Equation (7) can then be replaced by

$$K_w \nabla^2 h dt = a_w dh \qquad (8)$$

which is analogous to equation (4) for heat flow. We see then, that the quantity $a_h\rho$ representing quantity of heat flowing from a unit volume of conductor due to a unit change of temperature is analogous to the quantity a_w, representing the quantity of water drained from a unit volume of sand due to a unit change in head. We also see that this analogous relationship cannot be even approximately true unless the drawdown is of the order of the average capillary rise of the sand. Field investigations in connection with the rates of drainage of earth dams after reservoir drawdown, as well as laboratory studies in sands, have indicated that, within this limitation, equation (8) gives reasonable approximations of actual drainage rates.[6]

In water bearing sands of economic importance, the height of the capillary fringe is generally insignificant compared to the distance of drawdown. The yield pressure curve will look something like the dotted curve sketched in Figure 2. Beyond point A′, the quantity of water drainable from a unit volume of sand due to a unit change of head, *i.e.*, a_w in equation (8), is virtually zero. Then equation (8) becomes $\nabla^2 h = o$, which is the same as equation (2) for the steady state. This suggests that drainage of commercially important aquifers is a problem of steady-state, and not unsteady-state flow.

Now let us turn to a basic fallacy in present methods of estimating groundwater depletion by analogy with unsteady-state heat flow. Briefly, these methods use a basic equation as follows.[7]

$$K_w \nabla^2 h dt = G_s dh \qquad (9)$$

where G_s is the total quantity of water that can be drained from a unit volume of sand, or *specific yield*. This is represented graphically by the ordinate of the horizontal part of the dotted curve in Figure 2. It is generally determined experimentally by measuring the volume of water drained from a sand sample subjected to a centrifugal force of 1000 times gravity for two hours, divided by the volume of the sand sample. The right hand side of the equation must represent the net flux due to a head charge dh, or

$$dq_w/dh = G_s \qquad (10)$$

Hence, G_s must represent the slope of the linear yield pressure diagram. In terms of Figure 2, equations (9) and (10) require that the dotted curve be approximated by a line drawn from O to A at such a slope that the pressure head required to remove all drainable water be G_s into the volume of drainable water. No such actual relationship is apparent. Moreover, should the drawdown exceed an amount equivalent to the pressure head at A′, according to equation (9) still more water should be obtained as indicated by point A'_1. Actually, the ultimate quantity of water obtainable remains the same, and depends only on the specific yield of the sand. Therefore, the assumption that specific yield in ground water flow is analogous to the product of specific heat times density of conducting medium in heat flow appears untenable.

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6. Kellogg: Op. cit. pp 1269-1272.
7. Thies: Op. cit.

Comparison of field observations with computations based on equation (9) have shown surprisingly close agreement, *provided the line of observations passes through the center of the well.* The farther the line misses the center, the poorer the agreement. This is just what could be expected analytically. Any differential equation such as equation (9) must be solved so as to conform to certain boundary conditions. In the case of a pumped well, the requirement that head equal zero at the face of the well is a boundary condition as is the requirement that head equal a definite value at considerable distance from the well. Therefore, even if the differential equation be erroneous, it will give approximately correct results in the vicinity of the boundaries since the solution has been made to fit such boundaries. The farther away from the boundaries the point considered occurs, the greater will be the discrepancies due to the erroneous differential equation.

The following conclusions are drawn from the foregoing discussion: (1) assumption of analogy between unsteady state flows of heat and of groundwater is approximately correct only when drawdown does not exceed mean height of capillary rise, and hence, holds little promise for application to aquifers, (2) even within the limits for which such an analogy holds, the conception of specific yield as analogous to specific heat times density has no analytical foundation, and (3) drainage of aquifers appears from analytical considerations to be a problem of flow in the steady state.

Let us examine the last conclusion reached, to see if we can gain a physical picture of drainage in the steady state. We have defined steady state flow as that in which all outflow is replaced by an equal amount of inflow. At first glance, this does not seem to apply to drainage. Still, we can consider a type of drainage in which water flowing out is replaced by an equal amount of air flowing in. Thus, atmospheric pressure would exist in the pores of the sand immediately above the capillary water. This is in direct contrast to the condition of unsteady state flow, where we assume that the whole body is full of capillary water which is drained only when drawdown height exceeds height of capillary rise for a channel ending in a given pore space at the top of the sand. Under such a condition, there would be pressures less than an atmosphere in numerous pore spaces throughout the body at all times. If, however, the sand were sufficiently pervious, we could imagine a constant stream of air entering all pore spaces at the top of the sand from the moment drainage started. Then we would have a steady state flow in two phases, similar to that occurring when water follows oil through a sand.

Under such a condition of two-phase steady-state flow, we would be interested in the rate of advancement of the air-water interface. This is similar to the problem of rate of advancement of an oil-water interface as studied by Muskat.[8] His analysis makes analytical study of any but one dimensional flow very difficult, but facilitates study of models.

Other solutions for rate of depletion of water bearing sands have been proposed[9] neglecting capillarity or considering the sand to be governed by a single capillary force.

It is concluded therefore, that methods of estimating rates of depletion of aquifers on the basis of analysis of unsteady state flow, being based on a fallacious premise, holds little promise, and that the entire method of attack should be changed to an analysis of two-phase steady state flow. Investigations of drainage of actual sands on this basis are now in progress.

8. Muskat: *Flow of Homogenous Liquids,* McGraw- Hill, N. Y., 1937 p. 453 et seq.

9. Terzaghi: *Theoretical Soil Mechanics,* pp. 314-17, John Wiley and Sons, N. Y. 1943.